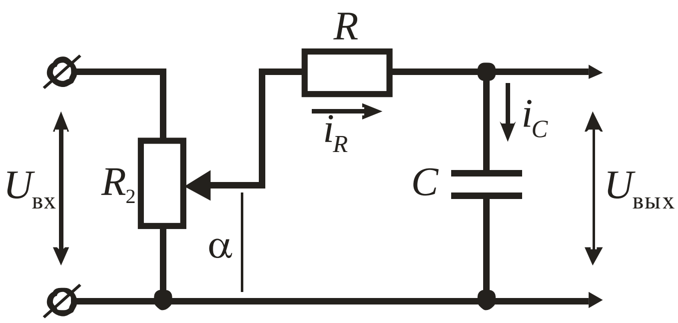
**Laboratory work 2**

**Studying of first- and second-order Typical Dynamic Elements parameters influence on their characteristics**

**2.1. Objective**

Building of Simulink models of *RC* and *RLC* two-port circuits, obtaining their time and frequency characteristics for various *R* and *C* values, and studying influence of the circuit element parameter values on the transient behavior.

**2.2 *RC* two-port circuit**




$U_{\text{in}} \to x$

$U_{\text{out}} \to y$

Figure 2.1 – Electrical schematic of an *RC* two-port circuit

Equations describing the circuit operation

$$\begin{cases} i_R = i_C \\ i_C = C\dfrac{dU_{out}(t)}{dt} \\ i_R = \dfrac{[\alpha U_{in}(t) - U_{out}(t)]}{R} \end{cases} \qquad (2.1)$$

The structural algorithmic diagram represents the graphical presentation of differential equations written in operator form.

The operator form of differential equations (2.1) is:

$$\begin{cases} i_R(p) = i_C(p) & (1') \\ i_C(p) = CpU_{out}(p) & (2') \\ i_R(p) = \dfrac{1}{R}[\alpha U_{in}(p) - U_{out}(p)] & (3') \end{cases} \qquad (2.2)$$

Transfer function is defined as ratio of output $U_{out}(p)$ and input $U_{in}(p)$ signals in operational form:

$$W(p)=\frac{y(p)}{x(p)}=\frac{U_{out}(p)}{U_{in}(p)}=\frac{\sum_{l=0}^{m} b_l p^l}{\sum_{k=0}^{m} a_k p^k} \qquad (2.3)$$

Let us obtain the expression linking these parameters by substituting equations (2') and (3') to equation (1'):

$$\frac{1}{R}[\alpha U_{in}(p)-U_{out}(p)]=CpU_{out}(p) \qquad (2.4)$$

The expression describing the output parameter is written on the left, while the expression describing the input parameter is on written on the right:

$$CpU_{out}(p)+\frac{1}{R}U_{out}(p)=\frac{1}{R}\alpha U_{in}(p), \qquad (2.5)$$

or:

$$[(C\cdot R)p+1]U_{out}(p)=\alpha U_{in}(p). \qquad (2.6)$$

Consequently, the transfer function of the *RC* circuit considered is

$$W(p)=\frac{U_{out}(p)}{U_{in}(p)}=\frac{\alpha}{(R\cdot C)p+1} \qquad (2.7)$$

The transfer function obtained shows that the considered *RC* two-port circuit is a first-order aperiodic dynamic element. Action of *R* and *C* values on its dynamic mode can be studied via Simulink model shown in Fig. 2.2.

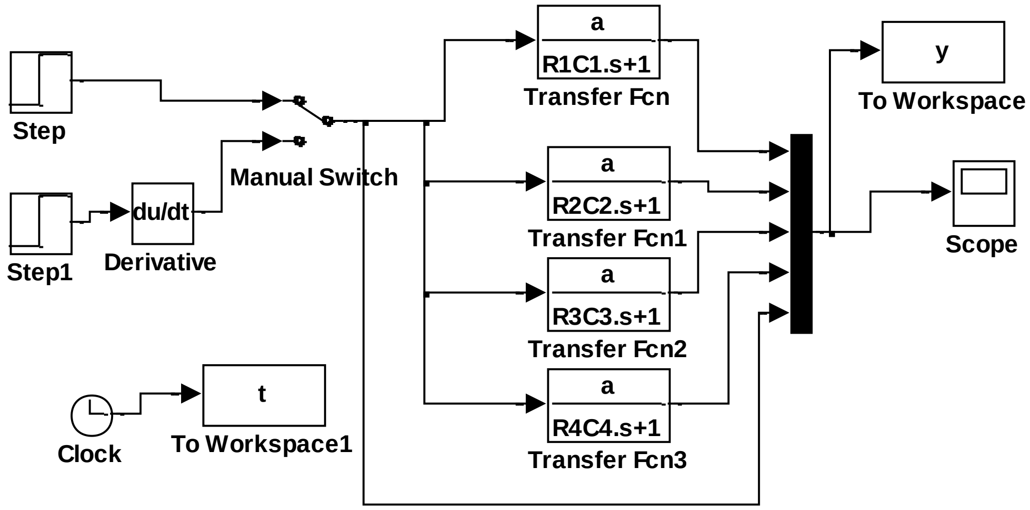
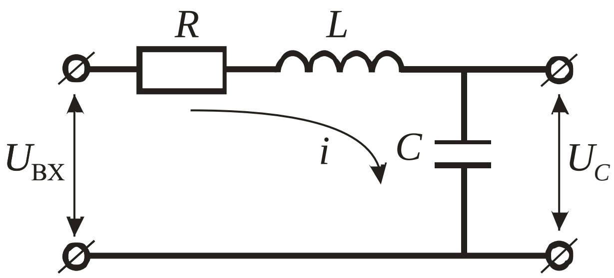



Figure 2.2 – Simulink model for studying dynamic behavior of *RC* two-port circuit versus resistance and capacitance values

### 2.3 *RLC* two-port circuit




$$U_{in} \to x$$
$$U_C \to y$$

Figure 2.3 - Electrical schematic of an *RCL* two-port circuit

Equations describing the circuit operation

$$\begin{cases} U_R = R \cdot i \\ U_C = \dfrac{1}{C}\int i \cdot dt \\ U_L = L \cdot \dfrac{di}{dt} \\ U_{in} = U_R + U_L + U_C = \\ \qquad = R \cdot i + L \cdot \dfrac{di}{dt} + \dfrac{1}{C}\int i \cdot dt \end{cases} \tag{2.8}$$

To obtain the expression containing only input voltage $U_{in}(t)$ and output voltage $U_C(t)$, let us write equations (2.8) in the operator form and make required transformations:

$$\begin{cases} U_R(p) = R \cdot i(p) & (1') \\ U_C(p) = \dfrac{1}{Cp} i(p) & (2') \\ U_L(p) = Lpi(p) & (3') \\ U_{in}(p) = U_R(p) + U_L(p) + U_C(p) = \\ \qquad = Ri(p) + Lpi(p) + \dfrac{1}{Cp} i(p) & (4') \end{cases} \tag{2.9}$$

From equation (2′), current $i(p)$ can be expressed with $U_C(p)$:

$$i(p) = CpU_C(p). \tag{2.10}$$

Substituting (2.10) to equations (1′), (3′) and (4′), we obtain

$$\begin{aligned} U_{in}(p) &= (R\cdot C)pU_C(p) + (L\cdot C)p^2U_C(p) + U_C(p) = \\ &= [(L\cdot C)p^2 + (R\cdot C)p + 1]U_C(p) \end{aligned} \tag{2.11}$$

The transfer function is ratio of output voltage $U_C(p)$ and input voltage $U_{in}(p)$ in operational form:

$$W(p) = \frac{U_C(p)}{U_{in}(p)} = \frac{1}{(L\cdot C)p^2 + (R\cdot C)p + 1} \tag{2.12}$$

The transfer function obtained shows that the considered *RLC* two-port circuit is a second-order dynamic element. Its transient mode behavior (either aperiodic or oscillatory) defining the type of the dynamic link depends on the parameters of the circuit elements.

Action of *R*, *L*, and *C* values on the circuit dynamic mode can be studied via Simulink model shown in Fig. 2.4.

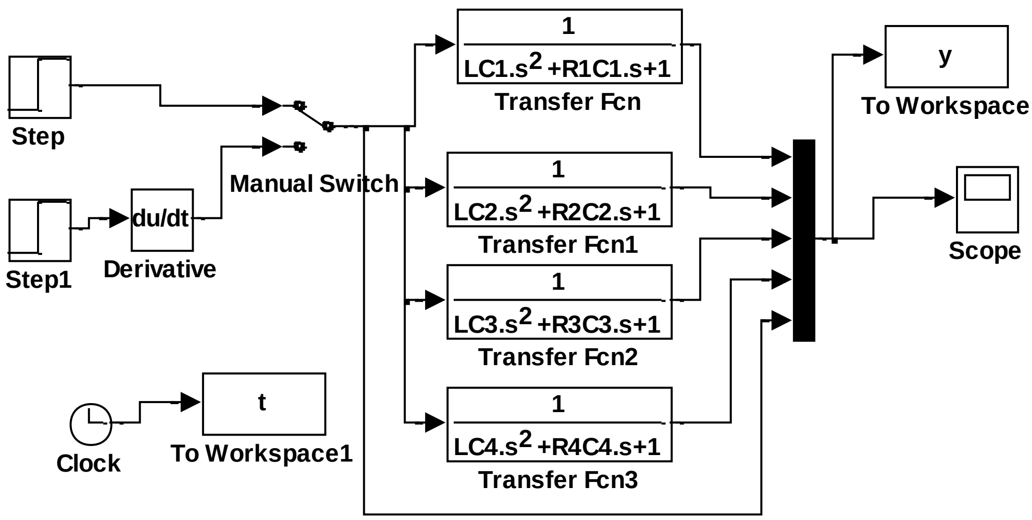



Figure 2.4 – Simulink model for studying dynamic behavior of *RLC* two-port circuit versus resistance, inductance and capacitance values

### 2.4 Assignment

Build Simulink models of *RC* and *RLC* circuits, obtain their time and frequency characteristics for different values of the circuit element parameters. Obtain their dynamic equations, transfer functions, and complex gains. Analyze dynamic behavior of the considered two-port *RC* and *RLC* circuits versus the circuit element parameter values specified in table 2.1 according to the variant.

Table 2.1 Variants of the two-port circuit element parameter values

| вариант | α | *L*, H | *R*1, Ω | *R*2, Ω | *R*3, Ω | *R*4, Ω | *C*1,F | *C*2,F | *C*3,F | *C*4,F |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.8 | 0.28 | 720 | 480 | 180 | 180 | $8.20 \cdot 10^{-5}$ | $8.20 \cdot 10^{-6}$ | 0.00047 | $8.20 \cdot 10^{-6}$ |
| 2 | 0.2 | 0.12 | 750 | 510 | 340 | 121 | $4.70 \cdot 10^{-5}$ | $4.70 \cdot 10^{-6}$ | $4.20 \cdot 10^{-6}$ | $8.20 \cdot 10^{-6}$ |
| 3 | 0.15 | 0.32 | 750 | 630 | 340 | 200 | $1.20 \cdot 10^{-6}$ | $1.20 \cdot 10^{-5}$ | $4.70 \cdot 10^{-6}$ | $8.20 \cdot 10^{-6}$ |
| 4 | 0.4 | 0.32 | 750 | 330 | 510 | 130 | $1.20 \cdot 10^{-6}$ | $1.20 \cdot 10^{-5}$ | $4.70 \cdot 10^{-6}$ | $8.20 \cdot 10^{-6}$ |
| 5 | 0.85 | 0.45 | 750 | 750 | 620 | 130 | $4.20 \cdot 10^{-5}$ | $8.20 \cdot 10^{-6}$ | $4.70 \cdot 10^{-6}$ | $6.20 \cdot 10^{-6}$ |
| 6 | 0.5 | 0.24 | 470 | 390 | 430 | 130 | $4.20 \cdot 10^{-5}$ | $1.20 \cdot 10^{-5}$ | $4.70 \cdot 10^{-6}$ | $8.20 \cdot 10^{-6}$ |
| 7 | 0.25 | 0.34 | 330 | 820 | 510 | 1580 | $2.20 \cdot 10^{-6}$ | $1.20 \cdot 10^{-5}$ | $4.70 \cdot 10^{-6}$ | $8.20 \cdot 10^{-6}$ |
| 8 | 0.1 | 0.12 | 330 | 470 | 100 | 300 | $2.20 \cdot 10^{-6}$ | $1.20 \cdot 10^{-5}$ | $4.70 \cdot 10^{-5}$ | $4.70 \cdot 10^{-5}$ |
| 9 | 0.9 | 0.85 | 510 | 470 | 430 | 820 | $8.20 \cdot 10^{-5}$ | $3.30 \cdot 10^{-5}$ | $1.80 \cdot 10^{-5}$ | $4.70 \cdot 10^{-5}$ |
| 10 | 0.75 | 0.55 | 470 | 470 | 360 | 680 | $4.20 \cdot 10^{-5}$ | $3.30 \cdot 10^{-6}$ | $1.80 \cdot 10^{-5}$ | $4.70 \cdot 10^{-5}$ |

## Appendix
## Plotting frequency characteristics for second-order TDE in Matlab

Dynamic equation of a second-order TDE is

$$T^2 \frac{d^2 y(t)}{dt^2} + 2\xi T \frac{dy(t)}{dt} = kx(t) \qquad (2.13)$$

and the transfer function is

$$W(s) = \frac{k}{T^2 s^2 + 2\xi Ts + 1}, \qquad (2.14)$$

where *k* is transfer coefficient; *T* is time constant; ξ is damping factor.

Value of damping factor ξ specifies the transient behavior of the TDE (fig. 2.5):
$\xi \geq 1$ is characteristic of aperiodic transient;
$\xi < 1$ is characteristic of oscillatory transient.

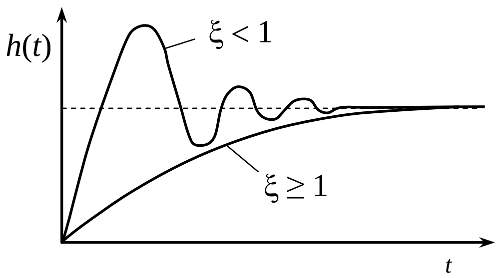



Figure 2.5 – Transient characteristics of 2nd –order TDE versus damping factor ξ

Unit step response (Heaviside response) is
$h(t) = k[1 - A\mathrm{e}^{-\alpha t}\sin(\omega t + \varphi)]$,
where *A* and φ are integration constants determined under zero initial conditions,
$A = 1/\sqrt{1-\xi^2}$; $\varphi = \arcsin\sqrt{1-\xi^2}$;
Unit impulse response () is
$w(t) = h'(t)$.

Time constant *T* and damping factor ξ can be found from frequency characteristics. Frequency characteristics are found using complex transfer function *W*(*j*ω)

$$W(j\omega) = W(s)\big|_{S=j\omega} = \frac{k}{T^2(j\omega)^2 + 2\xi T(j\omega) + 1} \qquad (2.15)$$

or

$$W(j\omega) = W(s)\big|_{S=j\omega} = \frac{k(1-T^2\omega^2)}{(1-T^2\omega^2)^2 + 4\xi^2 T^2\omega^2} - j\frac{2k\xi T\omega}{(1-T^2\omega^2)^2 + 4\xi^2 T^2\omega^2} \qquad (2.16)$$

**Second-order TDE (aperiodic transient)**

The initial parameters of the $2^{nd}$-order TDE are given in table 2.2.

Table 2.2 – Parameters of *RLC*-circuit (see fig. 2.3)

| *k* | *L*, H | *R*, Ω | *C*,F |
|---|---|---|---|
| 0.8 | 0.28 | 720 | $8.20 \cdot 10^{-5}$ |

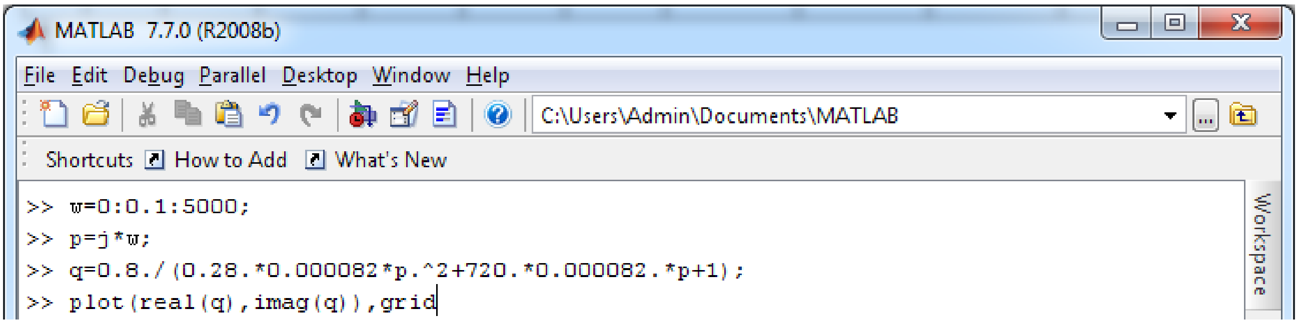
Transfer function of this dynamic element after substituting values of the *RLC*-circuit parameters to (2.12)

$$W(p) = W(s)\Big|_{s=p} = \frac{0.8}{0.28 \cdot 8.2 \cdot 10\wedge(-5)\, p^2 + 720 \cdot 8.2 \cdot 10\wedge(-5)\, p + 1}$$

According to (2.14), time constant *T* is equal to square root of the coefficient at $p^2$, that is $T=(0.28*8.2*10^{-5})^{0.5} = 0.0048$.

Damping factor ξ is equal to the coefficient at *p* divided by double time constant, that is $\xi=(720*8.2*10^{-5})/(2*0.0048)=6.15$, which is >1, therefore the *RLC*-circuit with the considered parameters behaves as a $2^{nd}$-order aperiodic TDE.

To plot frequency characteristics, the expression for the transfer function *W*(*p*) must be written in **Command window** and converted to complex transfer function *W*(*j*ω). For this, the following commands are introduced:




**Matlab** command **plot(real(q),imag(q)),grid** allows plotting amplitude-phase frequency characteristic:

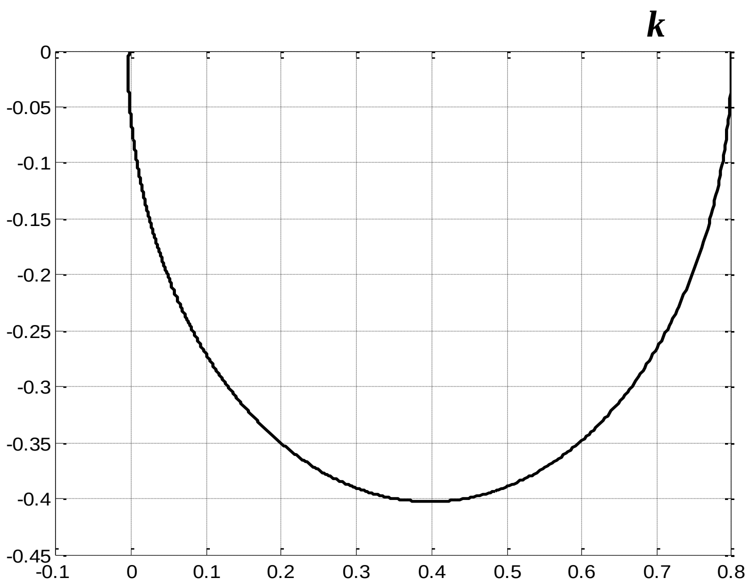



Figure 2.6 – Amplitude-phase frequency characteristic

Amplitude-phase frequency characteristic is the hodograph of the complex transfer function $W(j\omega)$.

The $W(j\omega)$ hodograph of a 2nd-order TDE starts at point $[k,0)]$ at $\omega = 0$, which allows determining the transfer coefficient $k$ in case the characteristic is obtained experimentally and the parameter is not known.

For the TDE studied, $k$ is 0.8 and the characteristic starts at point [0.8,0].

The hodograph (or amplitude-phase frequency characteristic) crosses *Im*-axis at point $[0, -k/(2\xi)]$. Hence it is also possible to determine damping factor $\xi$ from the amplitude-phase frequency characteristic

For the TDE studied, $-0.8/(2\xi) = -0.065$, therefore $\xi = 0.8/(2*0.065) = 6.15$, which exactly coincides with the computed value.

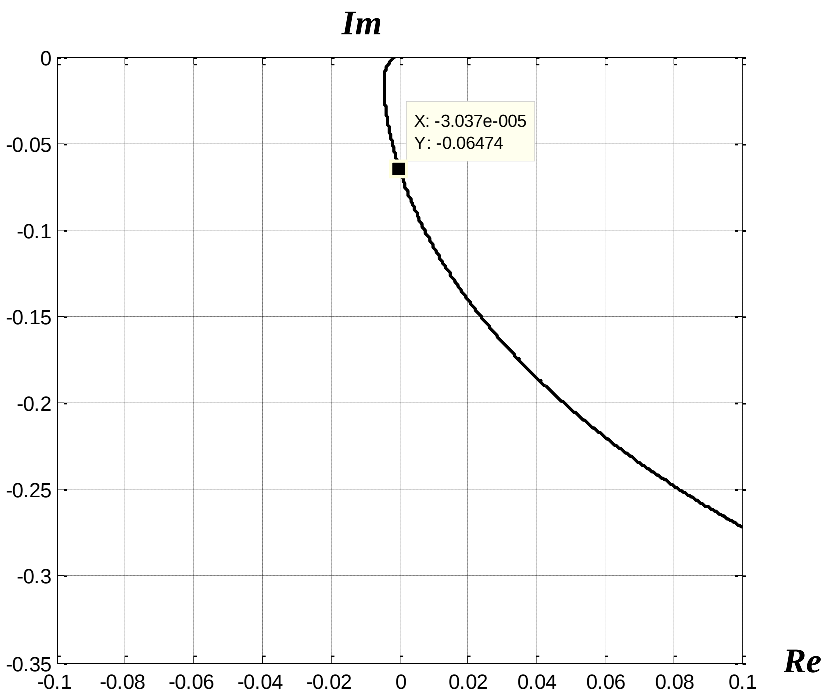
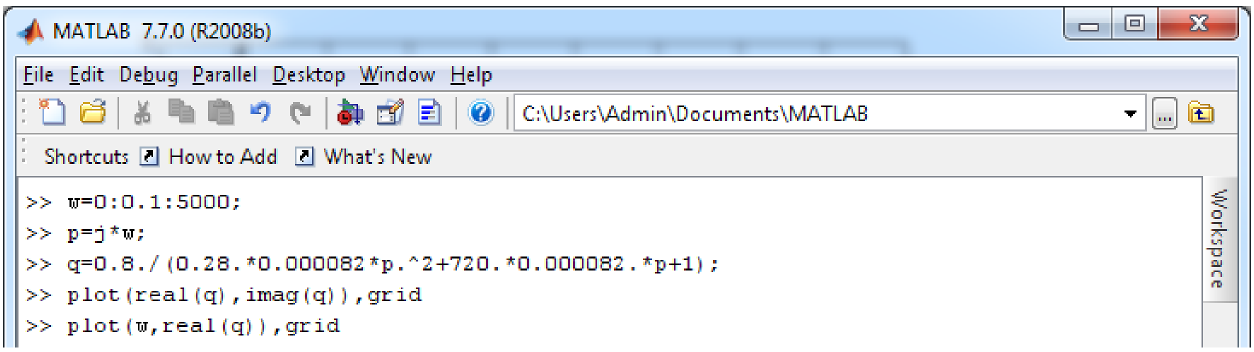



Figure 2.7 – Crosspoint of the amplitude-phase frequency characteristic and *Im*-axis

**Matlab** command **plot(w,real(q)),grid** allows plotting real frequency characteristic Re(ω).



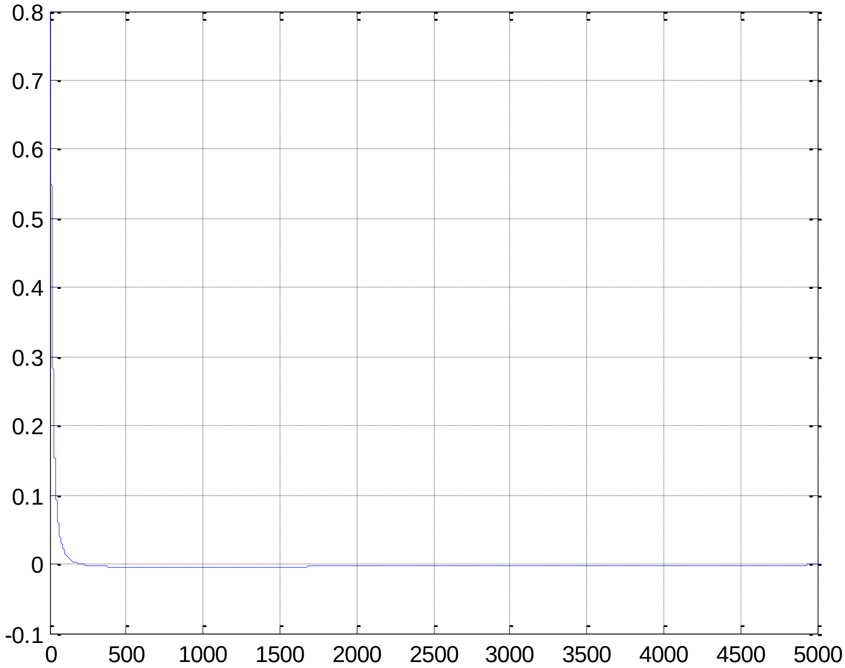



Figure 2.8 – Real frequency characteristic Re(ω)

Figure 2.9 shows the low-frequency domain of Re(ω).The real frequency characteristic of a 2nd-order TDE starts at $k$ at ω = 0, which makes it possible to determine the transfer coefficient of a 2nd-order TDE in case the parameter is unknown.

For the TDE studied the transfer coefficient, $k = 0.8$ and Re(ω) starts at 0.8.

At frequency $\omega = 1/T$, the real frequency characteristic of a 2nd-order TDE crosses ω–axis and, hence, the point of Re(ω) = 0 allows determining time constant $T$.

For the TDE studied, Re(ω) = 0 at frequency $\omega \approx 200\ s^{-1}$. Consequently, $T = 1/200 = 0.005$, which is in good agreement with the computed value 0.0048.

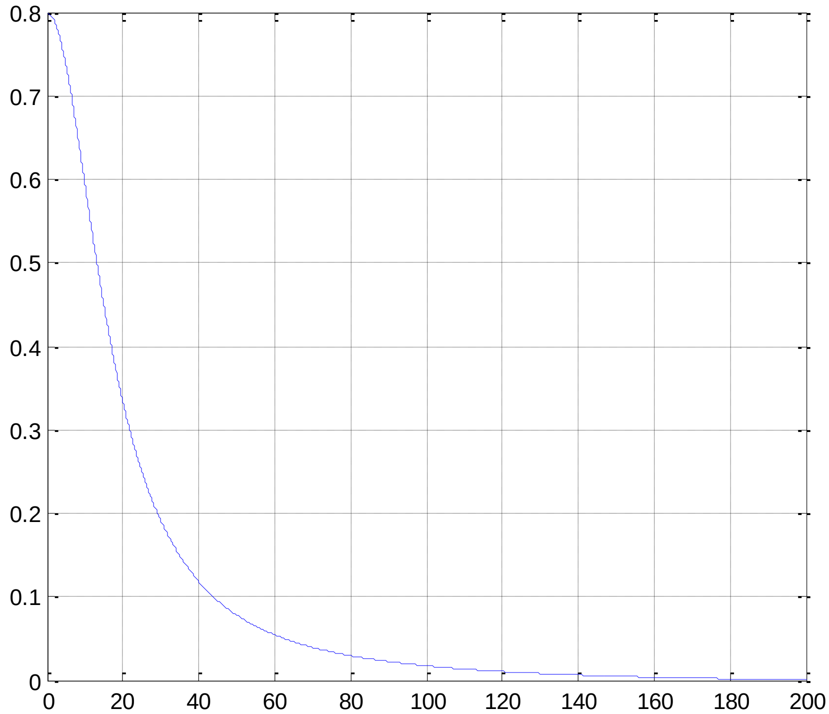
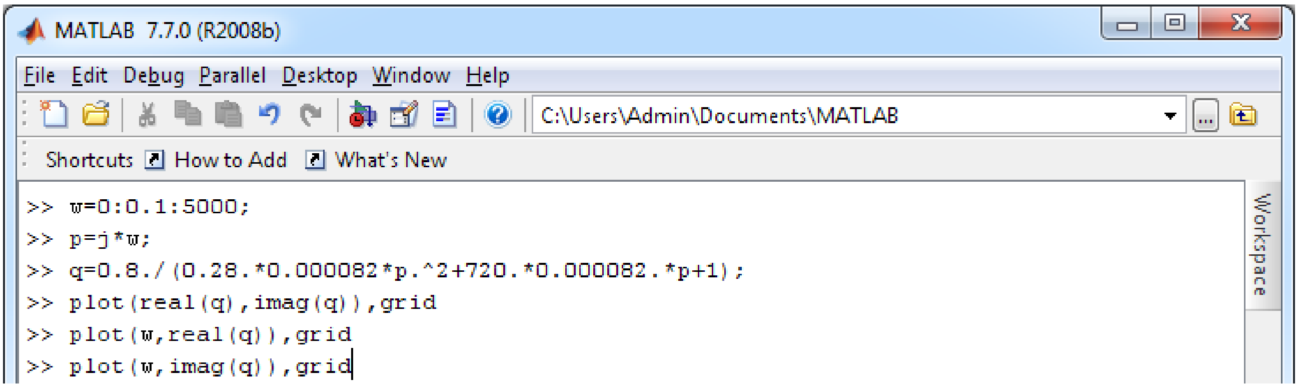



Figure 2.9 – Low-frequency domain of Re(ω)

**Matlab** command **plot(w,imag(q)),grid** allows plotting imaginary frequency characteristic Im(ω).



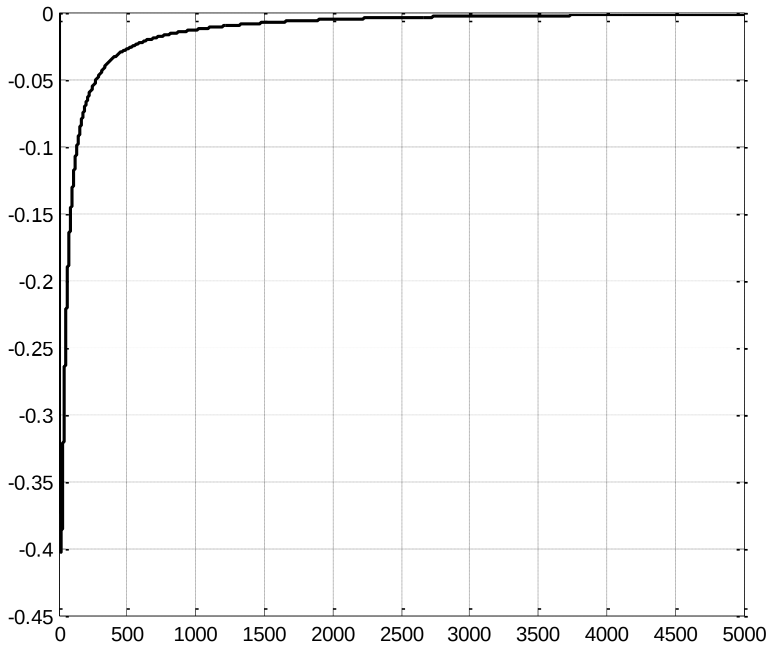




Figure 2.10 – Imaginary frequency characteristic Im(ω)

The characteristic helps determine time constant $T$ (if the damping factor $\xi$ is known) or damping factor $\xi$ (if the time constant is known) of a 2nd-order TDE.

At frequency $\omega = 1/T$, Im(ω) reaches the level of $-k/(2\xi)$.

Figure 2.11 presents the low-frequency domain to show minimum of Im(ω) and level of $-k/(2\xi)$.

For given $\xi$, according to the graph, at the level of -0.8/(2*6.15) = -0.065, frequency is approximately 220 $s^{-1}$, therefore $T$ = 1/220 = 0.0045, which is quite close to the computed value of 0.0048.

For given $T$, at frequency $\omega$ = 1/0.0048 = 208 $s^{-1}$, $-0.8/(2*\xi) \approx -0.067$ according to the graph. Therefore, $\xi$ = 0.8/(2*0.067) = 5.97 (≈6), which is in acceptable agreement with the computed value of 6.15 (≈6).

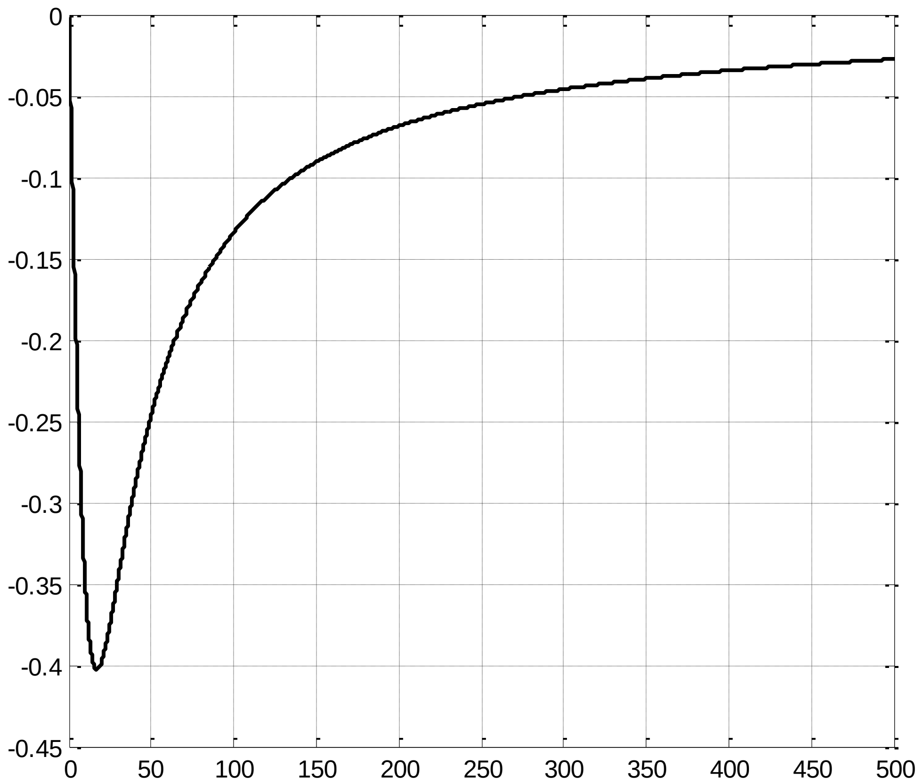
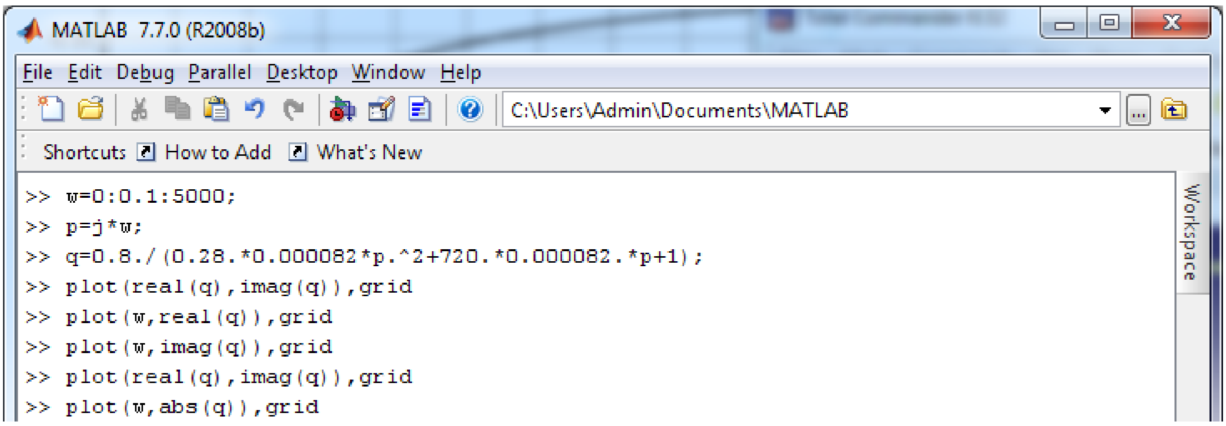


Figure 2.11 – Low frequency domain of Im(ω)

**Matlab** command **plot(w,abs(q)),grid** allows plotting magnitude frequency characteristic A(ω).




Magnitude frequency characteristic A(ω) helps determine transfer coefficient $k$ and time constant $T$ in case the characteristic is obtained experimentally and the parameters of the equipment that is modelled as a 2$^{nd}$-order TDE is unknown.

A(ω) starts at $k$, that is at $\omega = 0$, $A(0) = k$, which makes it possible to specify the transfer coefficient.

At $\omega = 1/T$, the curve declines to the level of $k/(2\xi)$ so time constant $T$ can be found.

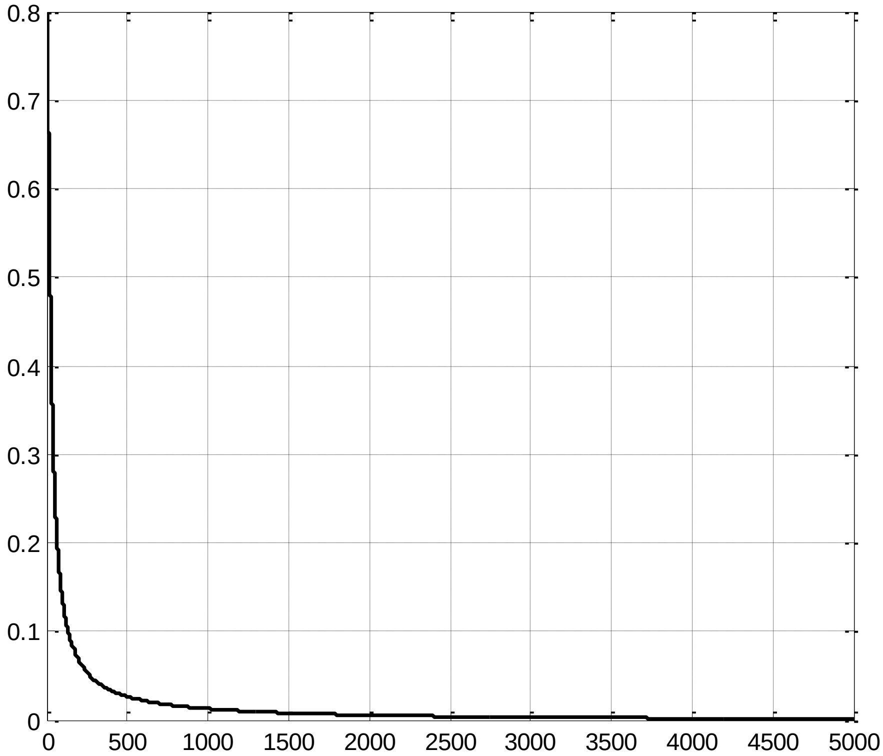


Figure 2.12 – Magnitude frequency characteristic A(ω)

Figure 2.13 presents the low-frequency domain of A(ω).

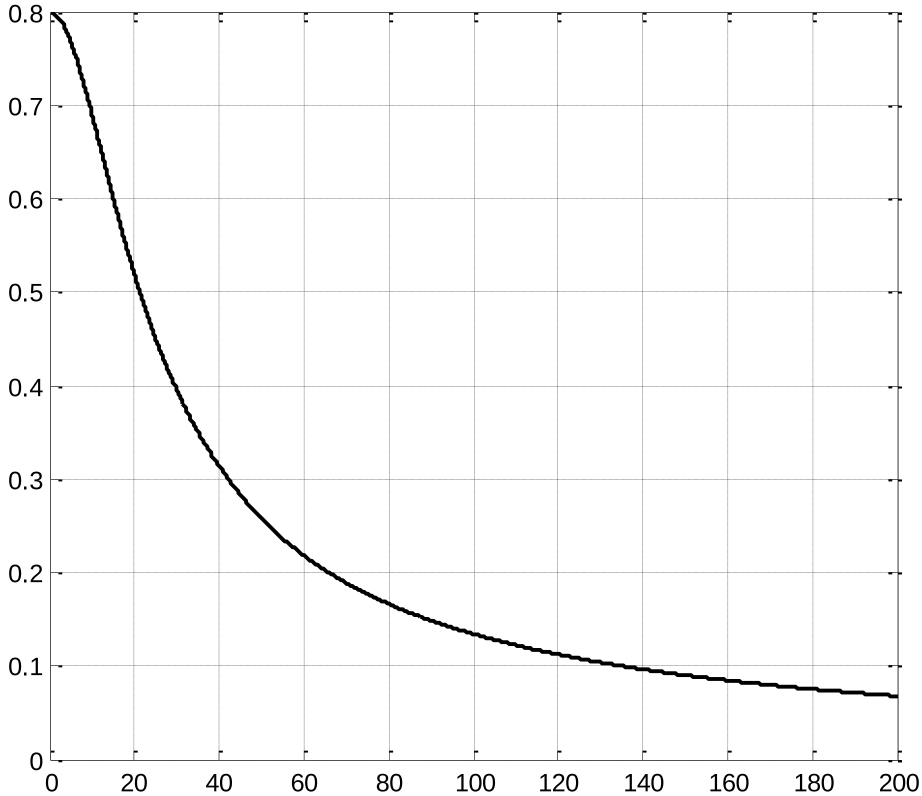


Figure 2.13 – Low frequency domain of A(ω)

For the TDE studied, at ω = 0 A(0) = 0.8.

At the level of $0.8/(2\cdot 6.15)$=0.065, ω≈200 $s^{-1}$. Consequently, $T$=1/200=0.005, which is quite close to the computed value 0.0048.

**Matlab** command **bode([$b_0$ $b_1$ … $b_m$],[$a_0$ $a_1$ … $a_m$]),grid** (where $b_0$ $b_1$ … $b_m$ and $a_0$ $a_1$ … $a_m$ are, correspondingly, coefficients of the numerator and denominator of the transfer function) allows plotting Bode magnitude $L(\omega)$ and phase $\varphi(\omega)$ plots.

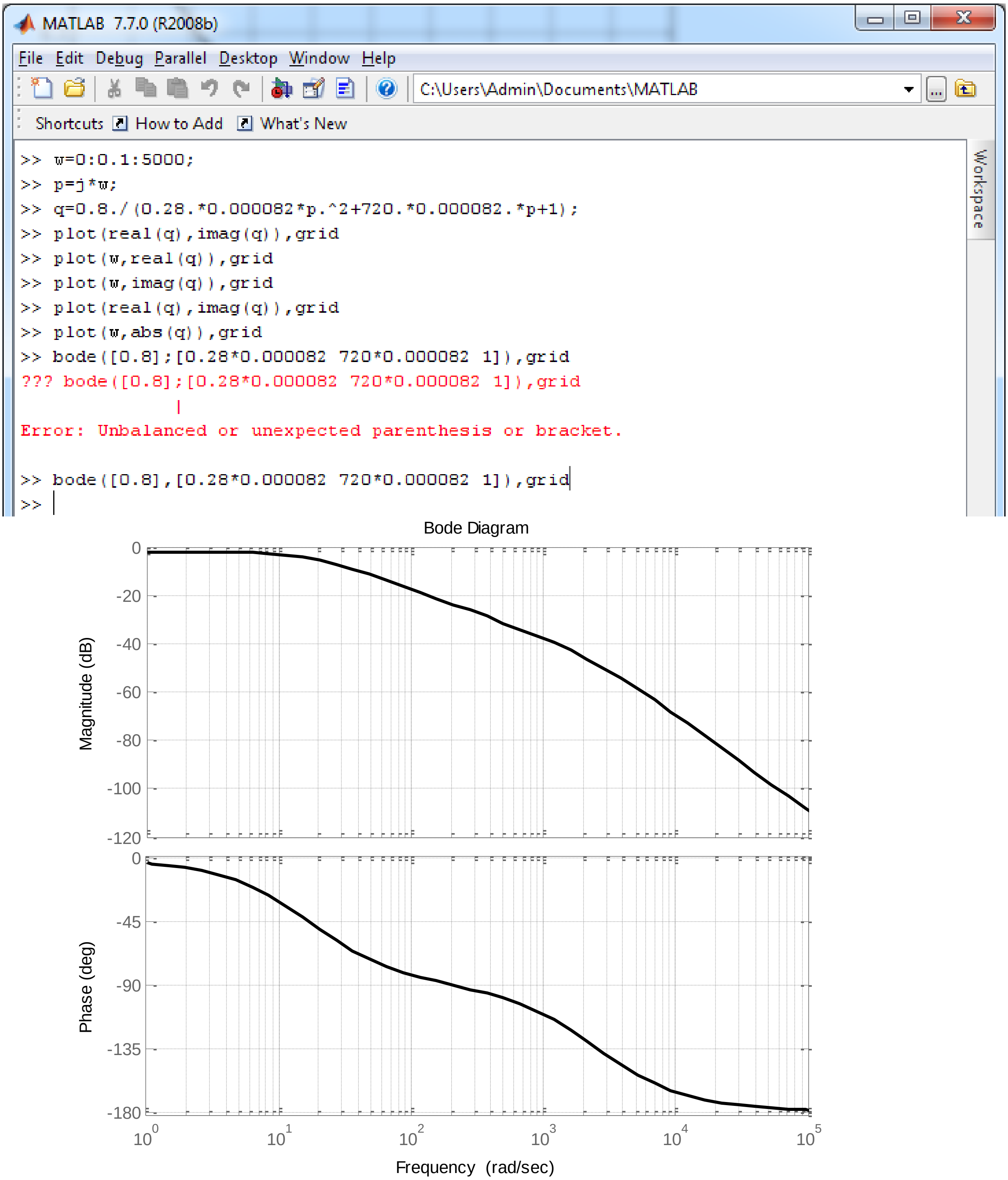



Figure 2.14 – Magnitude frequency characteristic A(ω)

The Bode magnitude plot demonstrates two slope breaks and the Bode phase plot has two waves, which is characteristic of a 2nd –order dynamic element with aperiodic transient and quite a high damping factor, which is the case considered.

The first break frequency $\omega_{b1}$ of L(ω) coincides with frequency $\omega_{-45}$ of φ(ω) crossing the level of -45°.

The other break frequency $\omega_{b2}$ of L(ω) coincides with frequency $\omega_{-135}$ of φ(ω) crossing the level of -135.

A 2nd-order aperiodic TDE (A-II TDE) is a complex dynamic element comprising two 1st-order aperiodic TDEs (A-I TDEs) connected in series, so the transfer function of A-II TDE (2.14) is product of transfer functions of the two A-I TDEs:

$$W(s)_{\text{A-II}} = W(s)_{1\text{A-I}} \cdot W(s)_{2\text{A-I}} = \frac{1}{T_1 s + 1} \cdot \frac{k}{T_2 s + 1}, \qquad (2.17a)$$

or

$$W(s)_{\text{A-II}} = \frac{k}{T^2 s^2 + 2\xi T s + 1} = \frac{1}{T_1 s + 1} \cdot \frac{k}{T_2 s + 1} = \frac{k}{(T_1 \cdot T_2) s^2 + (T_1 + T_2) s + 1}, \qquad (2.17b)$$

where $T_1$ is time coefficient of one 1st-order TDE and $T_2$ is time constant of the other 1st-order TDE.

From (2.17), $T_1$ and $T_2$ related to time constant $T$ and damping factor ξ of the 2nd-order TDE:

$T_1 \cdot T_2 = T$ and $T_1 + T_2 = 2\xi T$

$T_1$ and $T_2$ can be found from Bode plots as $T_1 = 1/\omega_{b1}$ and $T_2 = 1/\omega_{b2}$.

For the A-II TDE studied

$\omega_{b1} = \omega_{-45} \approx 18$, so $T_1 = 1/18 = 0.0556$

$\omega_{b2} = \omega_{-135} \approx 2400$, so $T_2 = 1/2400 = 0.000417$

$T_1 * T_2 = 0.0556*0.000417 = 0,00002318$

In the 2nd-order TDE studied, $T^2 = L*C = (0.28*8.2*10^{-5}) = 0.00002296$

$T_1 + T_2 = 0.0556 + 0.000417 = 0,056017$

In the 2nd-order TDE studied, $2\xi T = R*C = (720*8.2*10^{-5}) = 0.05904$

As we can see, results of the calculation and the simulation are in good agreement, which verifies the correctness of the characteristics plotted.

**Second-order TDE (oscillatory transient)**
The initial parameters of the $2^{nd}$-order TDE are given in table 2.3.

Table 2.3 – Parameters of the *RCL*-circuit (fig. 2.3)

| $k$ | $L$, H | $R1$, Ω | $C1$,F |
|---|---|---|---|
| 0.8 | 0.28 | 180 | $8.20 \cdot 10^{-6}$ |

Transfer function of this dynamic element after substituting values of the *RCL*-circuit parameters to (2.12) is

$$W(p) = W(s)\big|_{s=p} = \frac{0.8}{0.28 \cdot 8.2 \cdot 10\hat{}(-6)\,p^2 + 180 \cdot 8.2 \cdot 10\hat{}(-6)\,p + 1}$$

Time constant $T$ is equal to square root of the coefficient at $p^2$, that is
$T=(0.28*8.2*10^{-6})^{0.5} = 0.0015$.

Damping factor $\xi$ is equal to the coefficient at $p$ divided by double time constant, that is $\xi=(180*8.2*10^{-6})/(2*0.0015)=0.492$, which is <1, therefore the *RLC*-circuit with the considered parameters behaves as a $2^{nd}$-order oscillatory TDE.

To plot frequency characteristics, the above-described **Matlab** commands are gradually applied to the considered TDE.

```
>> w=0:0.1:5000;
>> p=j*w;
>> q=0.8./(0.28.*0.0000082*p.^2+180.*0.0000082.*p+1);
>> plot(real(q),imag(q)),grid
>> plot(w,real(q)),grid
>> plot(w,imag(q)),grid
>> plot(w,abs(q)),grid
>> bode([0.8],[0.28*0.0000082 180*0.0000082 1]),grid
```

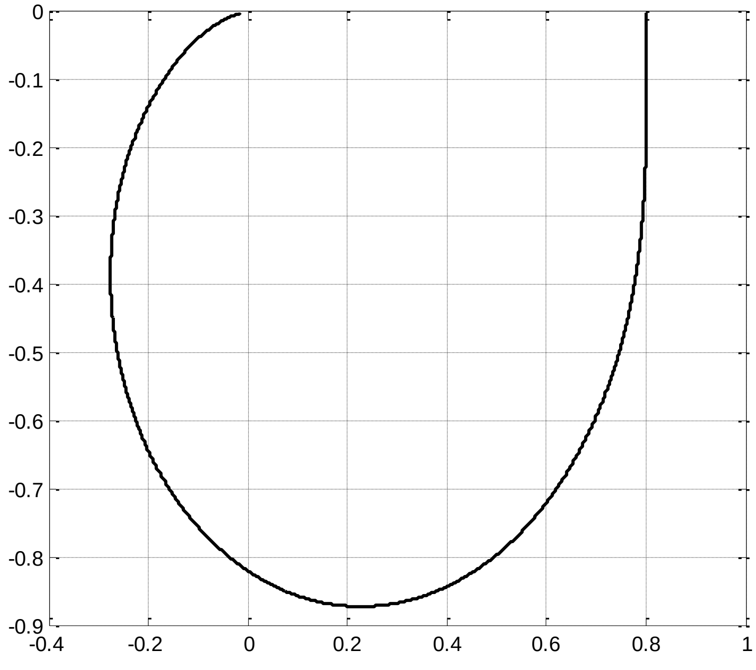



Figure 2.15 – Amplitude-phase frequency characteristic $W(j\omega)$

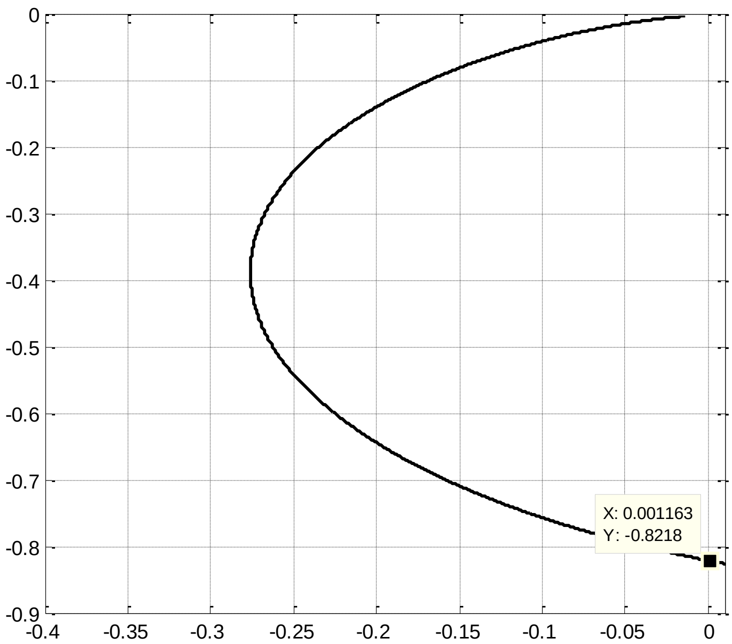



Figure 2.16 – Zoomed-up view of the amplitude-phase frequency characteristic crossing with *Im*-axis

-0.8/(2ξ)=-0.822, therefore damping factor ξ=0.8/(2*0.822)=0.49, which is equal to the computed value.

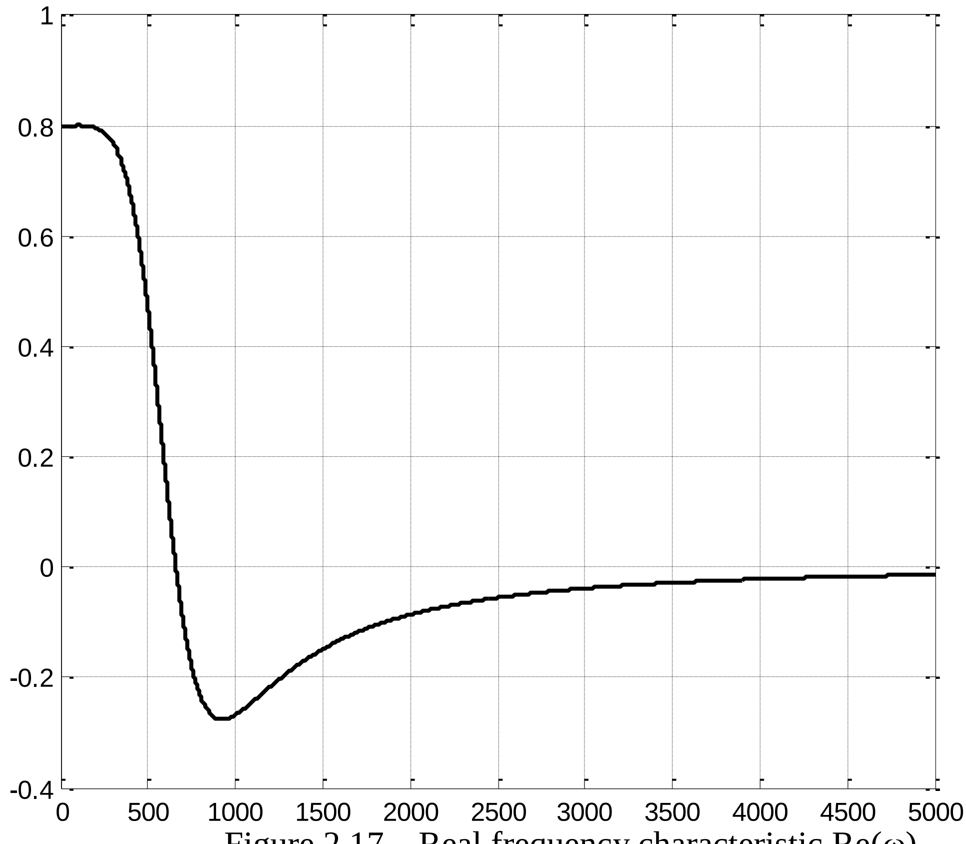



Figure 2.17 – Real frequency characteristic Re(ω)

At frequency ω=1/*T*, Re(ω) =0. The characteristic crosses ω–axis at frequency ω≈600 $s^{-1}$. Consequently, *T*=1/600=0.00167, which is close to the computed value 0.0015.

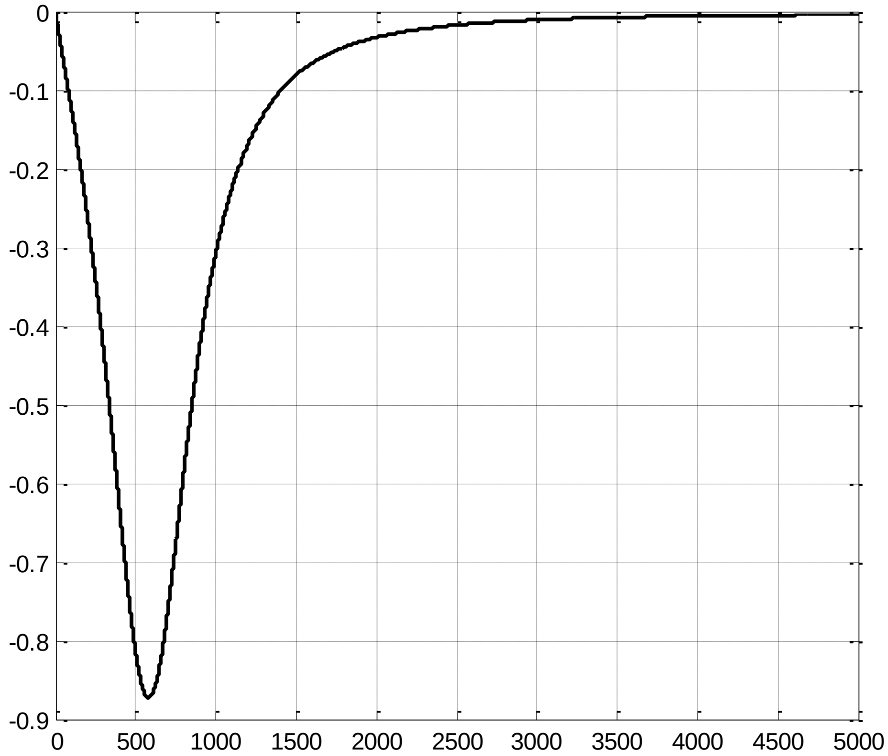



Figure 2.18 – Imaginary frequency characteristic Im(ω)

At frequency ω=1/*T*, Im(ω) crosses the level of -*k*/(2ξ).

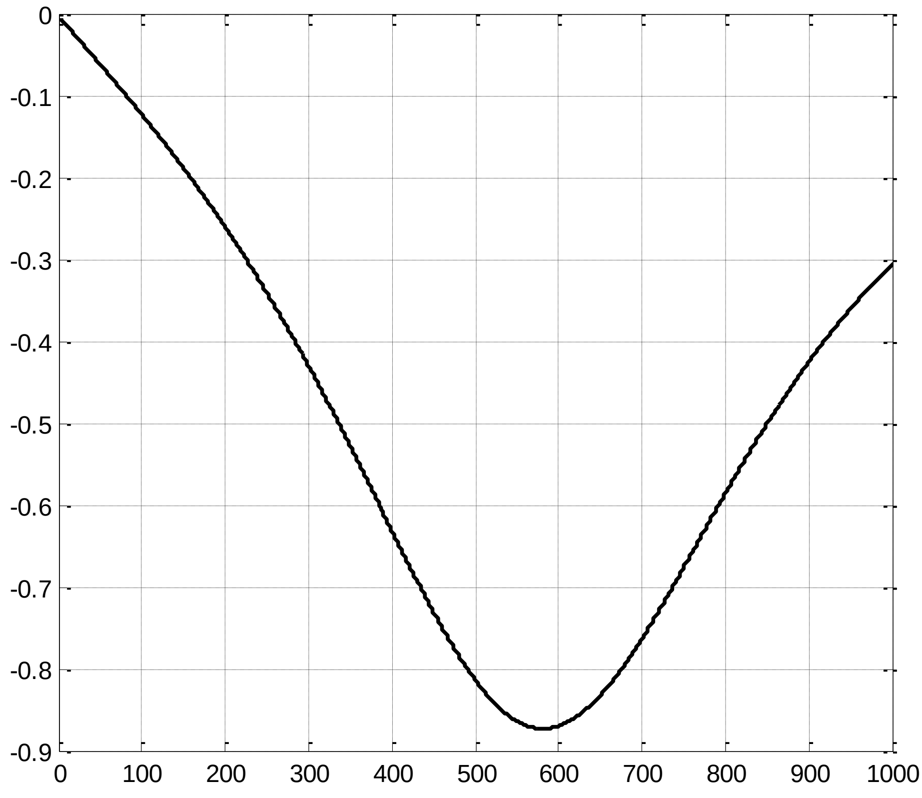



Figure 2.19 – Zoomed-up view of Im(ω) minimum

According to the graph, at the level of -0.8/(2*0.49)=-0.816, frequency is approximately 510 $s^{-1}$, therefore *T*=1/510=0.00196, which is quite close to the computed value of 0.0015.

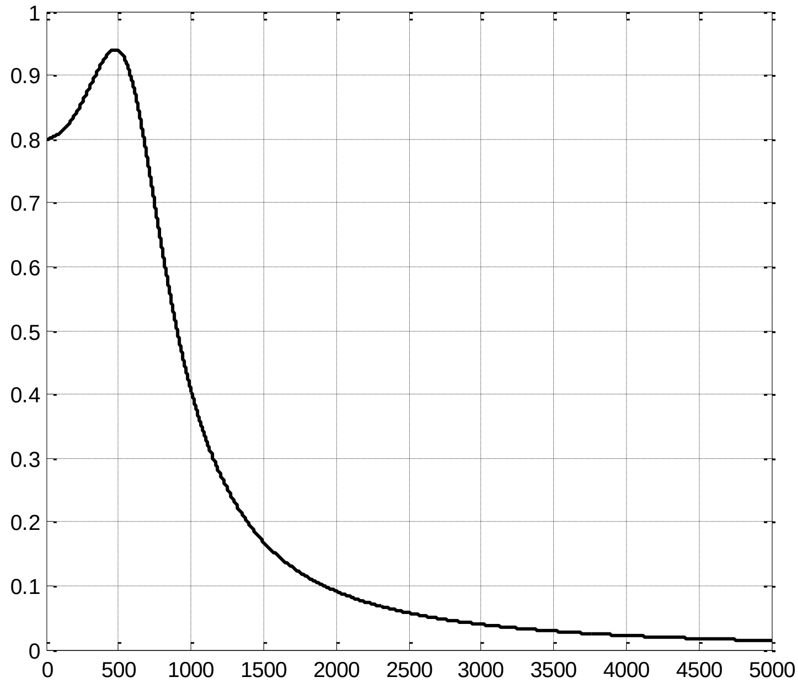



Figure 2.20 – Magnitude frequency characteristic A(ω)

The characteristic demonstrates a resonance, which is typical of a 2nd-order oscillatory DE (or a control system of the kind) with damping factor ξ<0.7. In the considered case, ξ=0.492. At the resonance frequency $\omega_{\text{res}} = \frac{\sqrt{1-2\xi^2}}{T}$, A(ω) reaches maximum of $A(\omega_{\text{res}}) = \frac{k}{2\xi\sqrt{1-\xi^2}}$ .

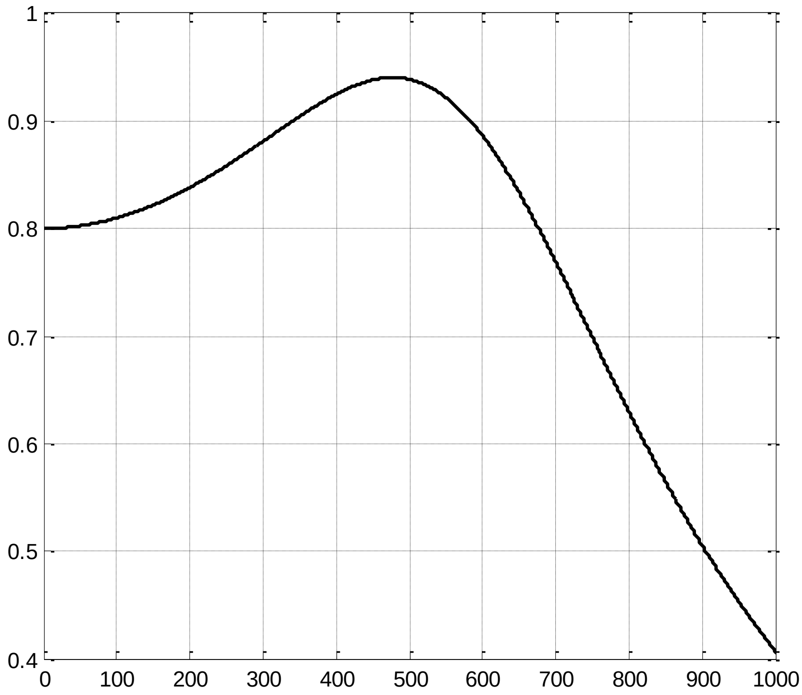



Figure 2.21 – Zoomed-up view of the A(ω) resonance

For the TDE studied, $\omega_{res}=\frac{\sqrt{1-2\cdot 0.49^2}}{0.0015}$ =480 s$^{-1}$ and $A(\omega_{res})=\frac{0.8}{2\cdot 0.49\sqrt{1-0.49^2}}$ =0.936. The results obtained by calculation completely coincide with the characteristic plotted.

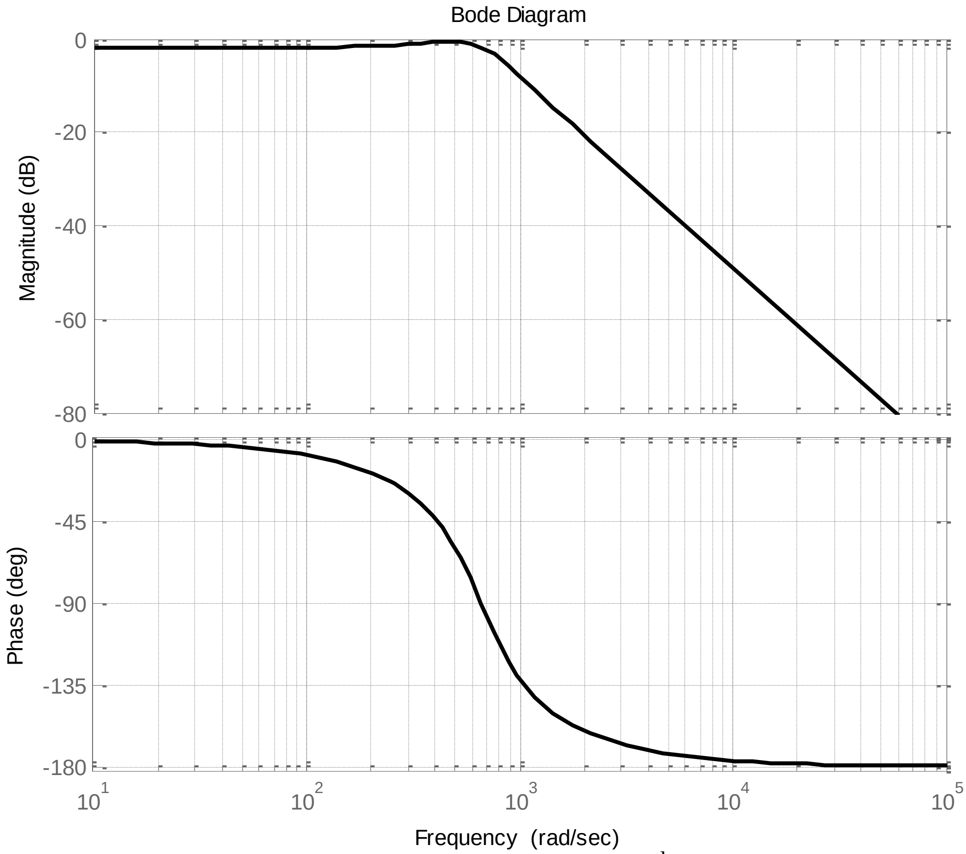



Figure 2.22 – Bode magnitude and phase plots for the 2$^{nd}$-order oscollatory TDE studied

Bode magnitude plot also shows resonance at the frequency $\omega_{res}\approx$500 s$^{-1}$. The slope of the curve breaks only once from 0 dB/decade to -40 dB/decade, which is characteristic of a 2$^{nd}$-order oscillatory TDE.

The break frequency $\omega_b \approx$ 630 s$^{-1}$ coincides with $\omega_{-90}$ which is the frequency of Bode phase plot crossing “-90” level, $\omega_b = \omega_{-90} \approx$ 630 s$^{-1}$.

Break frequency is opposite to time constant $T$, $\omega_b = 1/T$, therefore $T = 1/\omega_b$, and for the TDE studied $T$ = 1/630=0.00159 against 0.0015 computed, which verifies the correctness of the characteristics plotted.